Całki

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1. Wprost z definicji:

$$\int f'(x)dx = f(x) + C$$

$$\int dx = x + C$$

$$\int adx = ax + C$$

$$\int x^n dx = \begin{cases} \ln|x| + C; n = -1 \\ \frac{x^{n+1}}{n+1} + C; n \neq -1 \end{cases}$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin xdx = -\cos x + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$

$$\int e^x dx = e^x + C$$

$$\int \cos xdx = \sin x + C$$

$$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$$

$$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$$

$$\int \frac{dx}{1+x^2} = \operatorname{arctg} x + C$$

$$\int \sinh xdx = \cosh x + C$$

$$\int \cosh xdx = \sinh x + C$$

$$\int \frac{dx}{\cosh^2 x} = \operatorname{tgh} x + C$$

$$\int \frac{dx}{\sinh^2 x} = \operatorname{ctg} h + C$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \operatorname{arsinh} x + C$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arcosh} x + C$$

$$\int \frac{dx}{1-x^2} = \operatorname{artgh} x + C; x \in (-1,1)$$

$$\int \frac{dx}{x^2-1} = \operatorname{arctgh} x + C; x \in (-\infty,-1) \cup (1,\infty)$$

2. Wyprowadzenia:

1) $$\int \frac{dx}{a^2+x^2} = \frac{1}{a^2}\int \frac{dx}{1+(\frac{x}{a})^2} = \left\langle \begin{matrix} t=\frac{x}{a} \\ dt=\frac{dx}{a} \end{matrix} \right\rangle = \frac{1}{a}\int \frac{\frac{1}{a}dx}{1+(\frac{x}{a})^2} = \frac{1}{a}\int \frac{dt}{1+t^2} = \frac{1}{a}\arctan t + C = \frac{1}{a}\arctan\frac{x}{a} + C$$

2) założenia : $x \in R \setminus \bigcup_{k\in Z}\left\{k\pi + \frac{\pi}{2}\right\}$

$$\int \operatorname{tg} x dx = \int \frac{\sin x}{\cos x}dx = \left\langle \begin{matrix} t=\cos x \\ dt=-\sin x dx \end{matrix} \right\rangle = -\int \frac{-\sin x}{\cos x}dx = -\int \frac{dt}{t} = -\ln|t| + C = -\ln|\cos x| + C$$

3) założenia: $x \in R \setminus \bigcup_{k\in Z}\left\{k\pi\right\}$

$$\int \operatorname{ctg} x dx = \int \frac{\cos x}{\sin x}dx = \left\langle \begin{matrix} t=\sin x \\ dt=\cos x dx \end{matrix} \right\rangle = \int \frac{dt}{t} = \ln|t| + C = \ln|\sin x| + C$$

4) założenia : $x \in R_+$

$$\int \ln x dx = \left\langle \begin{matrix} u=\ln x \\ dv=1 \end{matrix} \middle| \begin{matrix} du=\frac{1}{x} \\ v=x \end{matrix} \right\rangle = x\ln x - \int \frac{1}{x}\cdot x dx = x\ln x - \int dx = x\ln x - x + C = x(\ln x - 1) + C$$

5) założenia: $x - a \neq 0$

$$\int \frac{Adx}{(x-a)^n} = A\int \frac{dx}{(x-a)^n} = \left\langle \begin{matrix} t=x-a \\ dt=dx \end{matrix} \right\rangle = A\int \frac{dt}{t^n} = \begin{cases} A\ln|t| + C; n=1 \\ \frac{A}{1-n}t^{1-n} + C; n\neq 1 \end{cases} = \begin{cases} A\ln|x-a| + C; n=1 \\ \frac{A}{1-n}(x-a)^{1-n} + C; n\neq 1 \end{cases}$$

6) założenia $ax + b \neq 0 ; a \cdot c \neq 0$

$$\int \frac{cx+d}{ax+b}dx = \frac{c}{a}\int \frac{x+\frac{d}{c}}{x+\frac{b}{a}}dx = \frac{c}{a}\int \frac{x+\frac{b}{a}+\frac{d}{c}-\frac{b}{a}}{x+\frac{b}{a}}dx = \frac{c}{a}\int (1+\frac{\frac{ad-bc}{ac}}{x+\frac{b}{a}})dx = \frac{c}{a}\int dx + \frac{c}{a}\cdot\frac{ad-bc}{ac}\int \frac{dx}{x+\frac{b}{a}} =$$

$$= \left\langle \begin{matrix} t=x+\frac{b}{a} \\ dt=dx \end{matrix} \right\rangle = \frac{c}{a}\int dx + \frac{ad-bc}{a^2}\int \frac{dt}{t} = \frac{c}{a}x + \frac{ad-bc}{a^2}\ln|t| + C = \frac{c}{a}x + \frac{ad-bc}{a^2}\ln\left|x+\frac{b}{a}\right| + C$$

7) założenia: $n \in Z \cap (3,\infty)$

$$\int \sin^n x dx = \int \sin^{n-1} x \sin x dx = \left\langle \begin{matrix} u=\sin^{n-1} x \\ dv=\sin x \end{matrix} \middle| \begin{matrix} du=(n-1)\sin^{n-2} x\cos x \\ v=-\cos x \end{matrix} \right\rangle =$$

$$-\sin^{n-1} x\cos x + \int (n-1)\sin^{n-2} x\cos^2 x dx = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\cos^2 x dx =$$

$$-\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x(1-\sin^2 x) x dx =$$

$$-\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx = \int \sin^n x dx$$

stąd:

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx$$

8) założenia $n \in Z \cap (3,\infty)$

$$\int \cos^n x dx = \int \cos^{n-1} x \cos x dx = \left\langle \begin{matrix} u = \cos^{n-1} x \\ dv = \cos x \end{matrix} \middle| \begin{matrix} du = -\sin x \cos^{n-2} x \\ v = \sin x \end{matrix} \right\rangle =$$

$$\sin x \cos^{n-1} x + \int (n-1)\cos^{n-2} x \sin^2 x dx = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)dx =$$

$$\sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x dx - +(n-1)\int \cos^n x dx$$

stąd:

$$\int \cos^n x dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

9) założenia: $a^2 - x^2 > 0$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a}\int \frac{dx}{\sqrt{1-(\frac{x}{a})^2}} = \left\langle \begin{matrix} t = \frac{x}{a} \\ dt = \frac{dx}{a} \end{matrix} \right\rangle = \int \frac{\frac{1}{a}dx}{\sqrt{1-(\frac{x}{a})^2}} = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin\frac{x}{a} + C$$

10) założenia: $a^2 - x^2 \geq 0$

$$\int \sqrt{a^2 - x^2} dx = \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx = a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} - \int x \frac{xdx}{\sqrt{a^2 - x^2}} =$$

$$\left\langle \begin{matrix} u = x \\ dv = \frac{x}{\sqrt{a^2 - x^2}} \end{matrix} \middle| \begin{matrix} du = 1 \\ v = -\sqrt{a^2 - x^2} \end{matrix} \right\rangle = a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} + x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2} dx$$

stąd:

$$\int \sqrt{a^2 - x^2} dx = \frac{a^2}{2}\int \frac{dx}{\sqrt{a^2 - x^2}} + \frac{x}{2}\sqrt{a^2 - x^2} = \langle 9 \rangle = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

11) założenia : $k > 0$

$$\int \frac{dx}{\sqrt{x^2 + k}} = \left\langle \begin{matrix} t = x + \sqrt{x^2 + k} \\ dt = (1 + \frac{x}{\sqrt{x^2 + k}})dx = \frac{x + \sqrt{x^2 + k}}{\sqrt{x^2 + k}} dx \\ \frac{dx}{\sqrt{x^2 + k}} = \frac{dt}{t} \end{matrix} \right\rangle = \int \frac{dt}{t} = \ln|t| + C = \ln|x + \sqrt{x^2 + k}| + C$$

12) założenia : $k > 0$

$$\int \sqrt{x^2 + k} dx = \int \frac{x^2 + k}{\sqrt{x^2 + k}} dx = \int x \frac{x}{\sqrt{x^2 + k}} dx + k\int \frac{dx}{\sqrt{x^2 + k}} = \left\langle \begin{matrix} u = x \\ dv = \frac{x}{\sqrt{x^2 + k}} \end{matrix} \middle| \begin{matrix} du = 1 \\ v = \sqrt{x^2 + k} \end{matrix} \right\rangle =$$

$$= k\int \frac{dx}{\sqrt{x^2 + k}} + x\sqrt{x^2 + k} - \int \sqrt{x^2 + k} dx = \int \sqrt{x^2 + k} dx$$

stąd:

$$\int \sqrt{x^2+k}dx = \frac{k}{2}\int \frac{dx}{\sqrt{x^2+k}} + \frac{x}{2}\sqrt{x^2+k} = \langle 11 \rangle = \frac{k}{2}\ln|x+\sqrt{x^2+k}| + \frac{x}{2}\sqrt{x^2+k} + C$$

13) założenia: $f(x) \neq 0$

$$\int \frac{f'(x)}{f(x)}dx = \left\langle \begin{array}{c} t = f(x) \\ dt = f'(x)dx \end{array} \right\rangle = \int \frac{dt}{t} = \ln|t| + C = \ln|f(x)| + C$$

14) założenia: $m \neq \pm n$

$$\int \sin mx \sin nx dx = \left[\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y) \quad \right] = \frac{1}{2}\int \cos[x(m-n)]dx - \frac{1}{2}\int \cos[x(m+n)]dx =$$

$$\left\langle \begin{array}{c} t = x(m-n) \\ \hline dt = (m-n)dx \end{array} \middle| \begin{array}{c} u = x(m+n) \\ \hline du = (m+n)du \end{array} \right\rangle =$$

$$\frac{1}{2(m-n)}\int \cos[x(m-n)](m-n)dx - \frac{1}{2(m+n)}\int \cos[x(m+n)](m+n)dx =$$

$$\frac{1}{2(m-n)}\int \cos tdt - \frac{1}{2(m+n)}\int \cos udu = \frac{\sin t}{2(m-n)} - \frac{\sin u}{2(m+n)} + C = \frac{\sin[x(m-n)]}{2(m-n)} - \frac{\sin[x(m+n)]}{2(m+n)} + C$$

15) założenia: $m \neq \pm n$

$$\int \cos mx \cos nx dx = \left[\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y) \quad \right] =$$

$$\frac{1}{2}\int \cos[x(m+n)]dx + \frac{1}{2}\int \cos[x(m-n)]dx = \left\langle \begin{array}{c} t = x(m+n) \\ \hline dt = (m+n)dx \end{array} \middle| \begin{array}{c} u = x(m-n) \\ \hline du = (m-n)dx \end{array} \right\rangle =$$

$$\frac{1}{2(m+n)}\int \cos[x(m+n)](m+n)dx + \frac{1}{2(m-n)}\int \cos[x(m-n)](m-n)dx =$$

$$\frac{1}{2(m+n)}\int \cos tdt + \frac{1}{2(m-n)}\int \cos udu = \frac{\sin t}{2(m+n)} + \frac{\sin u}{2(m-n)} + C = \frac{\sin[x(m+n)]}{2(m+n)} + \frac{\sin[x(m-n)]}{2(m-n)} + C$$

16) założenia: $m \neq \pm n$

$$\int \sin mx \cos nx dx = \left[\sin x \cos x = \frac{1}{2}[\sin(x+y) + \sin(x-y) \quad \right] = \frac{1}{2}\int \sin[x(m+n)]dx + \frac{1}{2}\int \sin[x(m-n)]dx =$$

$$\left\langle \begin{array}{c} t = x(m+n) \\ dt = (m+n)dx \end{array} \middle| \begin{array}{c} u = x(m-n) \\ du = (m-n)dx \end{array} \right\rangle =$$

$$\frac{1}{2(m+n)}\int \sin[x(m+n)](m+n)dx + \frac{1}{2(m-n)}\int \sin[x(m-n)](m-n)dx =$$

$$\frac{1}{2(m+n)}\int \sin tdt + \frac{1}{2(m-n)}\int \sin udu = -\frac{\cos t}{2(m+n)} =$$

$$-\frac{\cos u}{2(m-n)} + C = -\frac{\cos[x(m+n)]}{2(m+n)} - \frac{\cos[x(m-n)]}{2(m+n)} + C$$

17) założenia : $m \neq 0$

$$\int \cos mx dx = \left\langle \begin{matrix} t = mx \\ dt = mdx \end{matrix} \right\rangle = \frac{1}{m}\int \cos mx \cdot mdx = \frac{1}{m}\int \cos t dt = \frac{\sin t}{m} + C = \frac{\sin mx}{m} + C$$

18) założenia : $m \neq 0$

$$\int \sin mx dx = \left\langle \begin{matrix} t = mx \\ dt = mdx \end{matrix} \right\rangle = \frac{1}{m}\int \sin mx \cdot mdx = \frac{1}{m}\int \sin t dt = -\frac{\cos t}{m} + C = -\frac{\cos mx}{m} + C$$

19) założenia : $m \neq 0; mx \neq k\pi + \frac{\pi}{2}; k \in Z$

$$\int \operatorname{tg} mx dx = \int \frac{\sin mx}{\cos mx} dx = \left\langle \begin{matrix} t = \cos mx \\ dt = -m \sin mx \end{matrix} \right\rangle = -\frac{1}{m}\int \frac{-m \sin mx}{\cos mx} dx = -\frac{1}{m}\int \frac{dt}{t} = -\frac{\ln|t|}{m} + C =$$

$$\frac{\ln|\cos mx|}{m} + C$$

20) założenia : $m \neq 0; mx \neq k\pi; k \in Z$

$$\int \operatorname{ctg} mx dx = \int \frac{\cos mx}{\sin mx} dx = \left\langle \begin{matrix} t = \sin mx \\ dt = m \cos mx dx \end{matrix} \right\rangle = \frac{1}{m}\int \frac{m \cos mx}{\sin mx} dx = \frac{1}{m}\int \frac{dt}{t} = \frac{\ln|t|}{m} + C$$

$$\frac{\ln|\sin mx|}{m} + C$$

21) $$\int (ax+b)^n dx = \left\langle \begin{matrix} t = ax+b \\ dt = adx \end{matrix} \right\rangle = \frac{1}{a}\int (ax+b)^n adx = \frac{1}{a}\int t^n dt = \begin{cases} \frac{\ln t}{a} + C; n = -1 \\ \frac{t^{n+1}}{a(n+1)} + C; n \neq -1 \end{cases} =$$

$$\begin{cases} \frac{\ln(ax+b)}{a} + C; n = -1 \\ \frac{(ax+b)^{n+1}}{a(n+1)} + C; n \neq -1 \end{cases}$$

22) założenia: $b \neq 0; a + b\cos x \neq 0$

$$\int \frac{\sin x}{a + b\cos x} dx = \left\langle \begin{matrix} t = a + b\cos x \\ dt = -b \sin x \end{matrix} \right\rangle = -\frac{1}{b}\int \frac{-b \cos x dx}{a + b\cos x} = -\frac{1}{b}\int \frac{dt}{t} = -\frac{1}{b}\ln|t| + C = -\frac{\ln|a + b\cos x|}{b} + C$$

23) założenia: $a, b \in R_+$

$$\int \sqrt{a+bx} dx = \left\langle \begin{matrix} t = a+bx \\ dt = bdx \end{matrix} \right\rangle = \frac{1}{b}\int \sqrt{a+bx} \cdot bdx = \frac{1}{b}\int \sqrt{t} dt = \frac{1}{b} \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3b} \cdot t^{\frac{3}{2}} + C =$$

$$\frac{2}{3b}\sqrt{(a+bx)^3} + C$$

24) założenia: $x^2 + px + q \neq 0; p \quad -4q < 0$

$$\int \frac{2x+p}{(x^2+px+q)^n} dx = \left\langle \begin{matrix} t = x^2 + px + q \\ dt = (2x+p)dx \end{matrix} \right\rangle = \int \frac{dt}{t^n} = \begin{cases} \ln|t| + C; n = 1 \\ \frac{t^{1-n}}{1-n} + C; n \neq 1 \end{cases} = \begin{cases} \ln|x^2 + px + q| + C; n = 1 \\ \frac{(x^2+px+q)^{1-n}}{1-n} + C; n \neq 1 \end{cases}$$

25) $\int \frac{xdx}{(x^2+1)^n} = \left\langle \begin{matrix} t = x^2+1 \\ dt = 2xdx \end{matrix} \right\rangle = \frac{1}{2}\int \frac{2xdx}{(x^2+1)^n} = \frac{1}{2}\int \frac{dt}{t^n} = \begin{cases} \frac{1}{2}\ln|t| + C; n=1 \\ \frac{t^{1-n}}{2(1-n)} + C; n \neq 1 \end{cases} = \begin{cases} \frac{1}{2}\ln|x^2+1| + C; n=1 \\ \frac{(x^2+1)^{1-n}}{2-2n} + C; n \neq 1 \end{cases}$

26) założenia: $n \in Z; n > 1$

$$\int \frac{dx}{(x^2+1)^n} = \int \frac{x^2+1-x^2}{(x^2+1)^n} dx = \int \frac{x^2+1}{(x^2+1)^n} - \int \frac{x^2 dx}{(x^2+1)^n}$$

Oznaczmy $I_n = \int \frac{dx}{(x^2+1)^n}$, więc $\langle 26.1 \rangle I_n = I_{n-1} - \int \frac{x^2 dx}{(x^2+1)^n}$

$$\int \frac{x^2 dx}{(x^2+1)^n} = \int x \frac{xdx}{(x^2+1)^n} = \left\langle \begin{matrix} u = x \\ dv = \frac{x}{(x^2+1)^n} \end{matrix} \middle| \begin{matrix} du = 1 \\ v = \frac{-1}{(2-2n)(x^2+1)^{n-1}} \end{matrix} \right\rangle =$$

$$\frac{-x}{(2-2n)(x^2+1)^{n-1}} + \int \frac{dx}{(2-2n)(x^2+1)^{n-1}} = \frac{-x}{(2-2n)(x^2+1)^{n-1}} + \frac{1}{(2-2n)} \int \frac{dx}{(x^2+1)^{n-1}} =$$

$$= \frac{-1}{2n-2} \cdot \frac{x}{(x^2+1)^{n-1}} + \frac{1}{(2-2n)} I_{n-1}$$

podstawiając do <26.1>:

$$\int I_n = I_{n-1} + \frac{1}{2n-2} \cdot \frac{x}{(x^2+1)^{n-1}} - \frac{1}{2-2n} I_{n-1}$$

a ponieważ: $I_n = \int \frac{dx}{(x^2+1)^n}$ to:

$$\int \frac{dx}{(x^2+1)^n} = \frac{1}{2n-2} \cdot \frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(x^2+1)^{n-1}}$$

27) założenie: $\Delta = p^2 - 4q < 0; n \in Z_+$

$$\int \frac{dx}{(x^2+px+q)^n} = \int \frac{dx}{(x^2+px+\frac{p^2}{4}+q-\frac{p^2}{4})^n} = \int \frac{dx}{[(x+\frac{p}{2})^2 - \frac{p^2-4q}{4}]^n} = \int \frac{dx}{[(x+\frac{p}{2})^2 - \frac{\Delta}{4}]^n} =$$

$$(\frac{-\Delta}{4})^n \int \frac{dx}{[(\frac{2x-p}{\sqrt{-\Delta}})^2+1]^n} = \left\langle \begin{matrix} t = \frac{2x-p}{\sqrt{-\Delta}} \\ dt = \frac{2dx}{\sqrt{-\Delta}} \end{matrix} \right\rangle = (\frac{-\Delta}{4})^n \cdot \frac{\sqrt{-\Delta}}{2} \int \frac{\frac{2dx}{\sqrt{-\Lambda}}}{[(\frac{2x-p}{\sqrt{-\Delta}})^2+1]^n} =$$

$$(\frac{-\Delta}{4})^{n-\frac{1}{2}} \int \frac{dt}{(t^2+1)^n}$$

tutaj należy skorzystać ze wzoru <26> (dla n>1).

28) Założenia: $n \in Z_+; p^2 - 4q < 0$

$$\int\frac{Ax+B}{(x^2+px+q)^n}dx=\frac{A}{2}\int\frac{2x+\frac{2B}{A}+p-p}{(x^2+px+q)^n}dx=\frac{A}{2}\int\frac{2x+p}{(x^2+px+q)^n}dx+\frac{A}{2}\int\frac{\frac{2B}{A}-p}{(x^2+px+q)^n}dx=$$

$$=\left\langle\begin{matrix} t=x^2+px+q \\ dt=(2x+p)dx\end{matrix}\right\rangle=\frac{A}{2}\int\frac{dt}{t^n}+\frac{A}{2}\cdot\frac{2B-Ap}{A}\int\frac{dx}{(x^2+px+q)^n}=$$

$$\begin{cases}\frac{A}{2}\ln|t|+(B-\frac{Ap}{2})\int\frac{dx}{(x^2+px+q)^n};n=1 \\ \frac{At^{n1-n}}{2(1-n)}(B-\frac{Ap}{2})\int\frac{dx}{(x^2+px+q)^n};n\neq 1\end{cases}=\langle 27\rangle=\left\langle\begin{matrix} u=\frac{2x-p}{\sqrt{4q-p^2}} \\ du=\frac{2dx}{\sqrt{4p-q^2}}\end{matrix}\right\rangle=$$

$$=\begin{cases}\frac{A}{2}\ln(x^2+px+q)+(B-\frac{Ap}{2})(\frac{\sqrt{4q-p^2}}{2})\,\mathrm{arctg}\frac{2x-p}{\sqrt{4q-p^2}}+C \\ \frac{A(x^2+px+q)^{1-n}}{2(1-n)}+(B-\frac{Ap}{2})(\frac{-\Delta}{4})^{n-\frac{1}{2}}\int\frac{du}{\left(u^2+1\right)^n}\end{cases}$$

tutaj zastosować wzór <26>

29) Założenia: $b\in R_+$

$$\int\frac{dx}{(x-k)^2+b}=\left\langle\begin{matrix} t=x-k \\ dt=dx\end{matrix}\right\rangle=\int\frac{dt}{t^2+b}=\langle 1\rangle=\frac{1}{\sqrt{b}}arctg\frac{t}{\sqrt{b}}+C=\frac{1}{\sqrt{b}}arctg\frac{x-k}{\sqrt{b}}+C$$

30) Założenia : $\sin x\neq 0$

$$\int\frac{dx}{\sin x}=\int\frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\int\frac{dx}{2\,\mathrm{tg}\frac{x}{2}\cos^2\frac{x}{2}}=\left\langle\begin{matrix} t=tg\frac{x}{2} \\ dt=\frac{dx}{\cos^2\frac{x}{2}}\end{matrix}\right\rangle=\int\frac{dt}{t}=\ln|t|+C=\ln|\mathrm{tg}\frac{x}{2}|+C$$

31) Założenia : $\cos x\neq 0$

$$\int\frac{dx}{\cos x}=\left[\cos x=\sin(\frac{\pi}{2}+x)\right]=\int\frac{dx}{\sin(\frac{\pi}{2}+x)}=\left\langle\begin{matrix} t=\frac{\pi}{2}+x \\ dt=dx\end{matrix}\right\rangle=\int\frac{dt}{\sin t}=\langle 30\rangle=\ln|\mathrm{tg}\frac{t}{2}|+C=$$

$$\ln|\mathrm{tg}(\frac{\pi}{4}+\frac{x}{2})|+C$$

32) Założenia: $\sin x\cos x\neq 0$

$$\int\frac{dx}{\sin x\cos x}=\int\frac{dx}{\frac{1}{2}\sin 2x}=\int\frac{2dx}{\sin 2x}=\left\langle\begin{matrix} t=2x \\ dt=2dx\end{matrix}\right\rangle=\int\frac{dt}{\sin t}=\langle 30\rangle=\ln|\mathrm{tg}\frac{t}{2}|+C=\ln|\mathrm{tg}\,x|+C$$

33) Założenia : $\sin x\cos x\neq 0$

$$\int\frac{dx}{\sin^2 x\cos^2 x}=\int\frac{(\sin^2 x+\cos^2 x)dx}{\sin^2 x\cos^2 x}=\int\frac{\sin^2 xdx}{\sin^2 x\cos^2 x}-\int\frac{-\cos^2 xdx}{\sin^2 x\cos^2 x}=\int\frac{dx}{\cos^2 x}-\int\frac{dx}{-\sin^2 x}=$$

$$\mathrm{tg}\,x-\mathrm{ctg}\,x+C$$

34) Założenia : $x\in R\setminus\bigcup_{k\in Z}\left\{\frac{\pi}{2}+k\pi\right\}$

$$\int \operatorname{tg}^2 xdx=\int\frac{\sin^2 xdx}{\cos^2 x}=\int\frac{(1-\cos^2 x)dx}{\cos^2 x}=\int\frac{dx}{\cos^2 x}-\int\frac{\cos^2 xdx}{\cos^2 x}=\operatorname{tg}x-x+C$$

35) Założenia : $x\in R\setminus\bigcup_{k\in Z}\left\{k\pi\right\}$

$$\int \operatorname{ctg}^2 xdx=\int\frac{\cos^2 xdx}{\sin^2 x}=\int\frac{(1-\sin^2 x)dx}{\sin^2 x}=\int\frac{dx}{\sin^2 x}-\int\frac{\sin^2 xdx}{\sin^2 x}=-\operatorname{ctg}x-x+C$$

36) założenia : $n\in Z;n>2;x\in R\setminus\bigcup_{k\in Z}\left\{\frac{\pi}{2}+k\pi\right\}$

$$\int \operatorname{tg}^n xdx=\int \operatorname{tg}^{n-2} x\cdot tg^2 xdx=\int \operatorname{tg}^{n-2} x\cdot\frac{\sin^2 x}{\cos^2 x}dx=\int \operatorname{tg}^{n-2} x\cdot\frac{1-\cos^2 x}{\cos\ \ x}=\int \operatorname{tg}^{n-2} x\cdot(\frac{1}{\cos^2 x}-1)dx=$$

$$\int \operatorname{tg}^{n-2} x\cdot\frac{dx}{\cos^2 x}-\int \operatorname{tg}^{n-2} xdx=\left\langle\begin{matrix} t=\operatorname{tg}x\\ dt=\dfrac{dx}{\cos^2 x}\end{matrix}\right\rangle=\int t^{n-2}dt-\int \operatorname{tg}^{n-2} xdx=\frac{1}{n-1}t^{n-1}-\int \operatorname{tg}^{n-2} xdx=$$

$$\frac{1}{n-1}\operatorname{tg}^{n-1}-\int \operatorname{tg}^{n-2} xdx$$

37) założenia: $n\in Z;n>2;x\in R\setminus\bigcup_{k\in Z}\left(\ \ k\pi\ \ \right\}$

$$\int \operatorname{ctg}^n xdx=\int \operatorname{ctg}^{n-2} x\cdot \operatorname{ctg}^2 xdx=\int \operatorname{ctg}^{n-2} x\cdot\frac{\cos^2 x}{\sin^2 x}dx=\int \operatorname{ctg}^{n-2} x\cdot\frac{1-\sin^2 x}{\sin^2 x}dx=$$

$$\int \operatorname{ctg}^{n-2} x\cdot(\frac{1}{\sin^2 x}-1)dx=\int \operatorname{ctg}^{n-2} x\cdot\frac{dx}{\sin^2 x}-\int \operatorname{ctg}^{n-2} xdx=\left\langle\begin{matrix} t=\operatorname{ctg}x\\ dt=\dfrac{-dx}{\sin^2 x}\end{matrix}\right\rangle=$$

$$-\int t^{n-2}dt-\int \operatorname{ctg}^{n-2}xdx=-\frac{1}{n-1}t^{n-1}-\int \operatorname{ctg}^{n-2} xdx=-\frac{1}{n-1}\operatorname{ctg}^{n-1}x-\int \operatorname{ctg}^{n-2} xdx$$

38) $$\int x\sin cxdx=\left\langle\begin{matrix} u=x\\ dv=\sin cx\end{matrix}\middle|\begin{matrix} du=1\\ v=-\dfrac{1}{c}\cos cx\end{matrix}\right\rangle=-\frac{x\cos cx}{c}+\frac{1}{c}\int\cos cxdx=\frac{\sin cx}{c^2}-\frac{x\cos x}{c}+C$$

39) założenia: $\sin cx\neq 0$

$$\int\frac{dx}{\sin cx}=\left\langle\begin{matrix} t=cx\\ dt=cdx\end{matrix}\right\rangle=\frac{1}{c}\int\frac{dt}{\sin t}=\langle 30\rangle=\frac{1}{c}\ln|\operatorname{tg}\frac{t}{2}|+C=\frac{1}{c}\ln|\operatorname{tg}\frac{cx}{2}|+C$$

40) założenia: $n\notin\{-2,-1\}$

$$\int x(ax+b)^n dx = \left\langle \begin{matrix} u = x \\ dv = (ax+b)^n \end{matrix} \middle| \begin{matrix} du = 1 \\ v = \dfrac{(ax+b)^{n+1}}{a(n+1)} \end{matrix} \right\rangle = \frac{x(ax+b)^{n+1}}{a(n+1)} - \int \frac{(ax+b)^{n+1} dx}{a(n+1)} =$$

$$\left\langle \begin{matrix} t = ax+b \\ dt = adx \end{matrix} \right\rangle = \frac{x(ax+b)^{n+1}}{a(n+1)} - \frac{1}{a^2(n+1)} \int t^{n+1} dt = \frac{x(ax+b)^{n+1}}{a(n+1)} - \frac{1}{a^2(n+1)} \cdot \frac{t^{n+2}}{n+2} + +C =$$

$$\frac{x(ax+b)^{n+1}}{a(n+1)} - \frac{(ax+b)^{n+2}}{a^2(n+1)(n+2)} + C = \frac{a(n+2)x(ax+b)^{n+1} - (ax+b^{n+2})}{a^2(n+1)(n+2)} + C =$$

$$(ax+b)^{n+1} \cdot \frac{a(n+2)x - ax - b}{a^2(n+1)(n+2)} + C = \frac{a(n-1)x - b}{a^2(n+1)(n+2)}(ax+b)^{n+1} + C$$

41) założenie : $ax+b \neq 0$

$$\int \frac{xdx}{ax+b} = \left\langle \begin{matrix} t = ax+b \\ dt = adx \\ x = \dfrac{t-b}{a} \end{matrix} \right\rangle = \frac{1}{a} \int \frac{(t-b)dt}{at} = \frac{1}{a^2} \int dt - \frac{b}{a^2} \int \frac{dt}{t} = \frac{t}{a^2} - \frac{b}{a^2} \ln|a| + C = \frac{ax+b}{a^2} - \frac{b}{a^2} \ln|ax+b| + C =$$

$$\frac{x}{a} + \frac{b}{a^2} - \frac{b}{a^2} \ln|ax+b| + C$$

Ponieważ $\frac{b}{a^2}$ jest stałą można ją włączyć do stalej całkowania. Tak więc:

$$\int \frac{xdx}{ax+b} = \frac{x}{a} - \frac{a}{b^2} \ln|ab+b| + C$$

42) założenia: $ax+b \neq 0$

$$\int \frac{xdx}{(ax+b)^2} = \left\langle \begin{matrix} t = ax+b \\ dt = adx \\ x = \dfrac{t-b}{a} \end{matrix} \right\rangle = \frac{1}{a} \int \frac{(t-b)^2 dt}{at^2} = \frac{1}{a^2} \int \frac{dt}{t} - \frac{1}{a^2} \int \frac{dt}{t^2} = \frac{1}{a^2} \ln|t| - \frac{1}{a^2} \cdot \frac{1}{-t} + C =$$

$$\frac{1}{a^2} \ln|t| + \frac{1}{a^2 t} = \frac{1}{a^2(ax+b)} + \frac{1}{a^2} \ln|ax+b| + C$$

43) założenia: $ax + b \neq 0; n \notin \{1,2\}$

$$\int \frac{xdx}{(ax+b)^n} = \left\langle \begin{matrix} t = ax+b \\ dt = adx \\ x = \dfrac{t-b}{a} \end{matrix} \right\rangle = \frac{1}{a} \int \frac{(t-b)dt}{at^t} = \frac{1}{a^2} \int t^{1-n} dt - \frac{b}{a^2} \int t^{-n} = \frac{1}{a^2} \cdot \frac{t^{2-n}}{2-n} - \frac{b}{a^2} \cdot \frac{t^{1-n}}{1-n} + C =$$

$$t^{1-n} \cdot \frac{t(1-n) - b(2-n)}{a^2(2-n)(1-n)} - +C = \frac{(ax+b)(1-n) - 2b + bn}{a^2(2-n)(1-n)(ax+b)^{n-1}} + C = \frac{ax - nax + b - bn - 2b + bn}{a^2(2-n)(1-n)(ax+b)^{n-1}} + C =$$

$$\frac{a(1-n)x - b}{a^2(2-n)(1-n)(ax+b)^{n-1}} + C = \frac{a(1-n)x - b}{a^2(n-1)(n-2)(ax+b)^{n-1}} + C$$

44) założenia: $ax + b \neq 0$

$$\int \frac{x^2 dx}{ax+b} = \left\langle \begin{matrix} t = ax+b \\ dt = adx \\ x = \dfrac{t-b}{a} \end{matrix} \right\rangle = \frac{1}{a}\int \frac{(t-b)^2 dt}{a^2 t} = \frac{1}{a^3}\int \frac{t^2 - 2tb + b^2}{t} dt = \frac{1}{a^3}\left[\int tdt - 2b\int dt + b^2 \int \frac{dt}{t}\right] =$$

$$\frac{1}{a^3}\left[\frac{t^2}{2} - 2bt + b^2 \ln|t|\right] + C = \frac{1}{a^3}\left[\frac{(ax+b)^2}{2} - 2b(ax+b) + b^2 \ln|ax+b|\right] + C$$

45) założenia: $ax + b \neq 0$

$$\int \frac{x^2 dx}{(ax+b)^2} = \left\langle \begin{matrix} t = ax+b \\ dt = adx \\ x = \dfrac{t-b}{a} \end{matrix} \right\rangle = \frac{1}{a}\int \frac{(t-b)^2 dt}{a^2 t^2} = \frac{1}{a^3}\int \frac{t^2 - 2tb + b^2}{t^2} dt = \frac{1}{a^3}\left[\int dt - 2b\int \frac{dt}{t} + b^2 \int \frac{dt}{t^2}\right] =$$

$$\frac{1}{a^3}\left[t - 2b\ln|t| - \frac{b^2 t}{}\right] + C = \frac{1}{a^3}\left[ax + b - 2b\ln|ax+b| - \frac{b^2}{ax+b}\right] + C$$

46) założenia: $ax + b \neq 0$

$$\int \frac{x^2 dx}{(ax+b)^3} = \left\langle \begin{matrix} t = ax+b \\ dt = adx \\ a = \dfrac{t-b}{a} \end{matrix} \right\rangle = \frac{1}{a}\int \frac{(t-b)^2 dt}{a^2 t^3} = \frac{1}{a^3}\int \frac{t^2 - 2tb + b^2}{t^3} dt = \frac{1}{a^3}\left[\int \frac{dt}{t} - 2b\int \frac{dt}{t^2} + b^2 \int \frac{dt}{t^3}\right] =$$

$$\frac{1}{a^3}\left[\ln|t| + \frac{2b}{t} - \frac{b^2}{2t^2}\right] + C = \frac{1}{a^3}\left[\ln|ax+b| + \frac{2b}{ax+b} - \frac{b^2}{2(ax+b)^2}\right] + C$$

47) założenia: $ax + b \neq 0; n \notin \{1,2,3\}$

$$\int \frac{x^2 dx}{(ax+b)^n} = \left\langle \begin{matrix} t = ax+b \\ dt = adx \\ x = \dfrac{t-b}{a} \end{matrix} \right\rangle = \frac{1}{a}\int \frac{(t-b)^2 dt}{a^2 t^n} = \frac{1}{a^3}\int \frac{t^2 - 2bt + b^2}{t^n} = \frac{1}{a^3}\left[\int t^{2-n} dt - 2b\int t^{1-n} dt + b^2 \int t^{-n} dt\right] =$$

$$\frac{1}{a^3}\left[\frac{t^{3-n}}{3-n} - \frac{2bt^{2-n}}{2-n} + \frac{b^2 t^{1-n}}{1-n}\right] + C = \frac{1}{a^3}\left[-\frac{1}{(n-3)(ax+b)^{n-3}} + \frac{2b}{(n-2)(ax+b)^{n-2}} - \frac{b^2}{(n-1)(ax+b)^{n-1}}\right] + C$$

48) założenia: $x(ax+b) \neq 0; b \neq 0$

$$\int \frac{dx}{x(ax+b)}$$

Funkcję podcałkową rozbijam na sumę ułamków prostych:

$$\frac{1}{x(ax+b)} \equiv \frac{A}{x} + \frac{B}{ax+b}$$

$$1 \equiv A(ax+b) + Bx$$

Podstawiam $x = 0$:

$$1 \equiv Ab \Rightarrow A = \frac{1}{b}$$

Postawiam $x = -\frac{b}{a}$

$$1 \equiv B \cdot \left(-\frac{b}{a}\right) \Rightarrow B = -\frac{a}{b}$$

Tak wię a:

$$\int \frac{dx}{x(ax+b)} = \frac{1}{b}\int \frac{dx}{x} - \frac{a}{b}\int \frac{dx}{ax+b} =< 21 >= \frac{1}{b}\ln|x| - \frac{1}{b}\ln|ax+b| + C = \ln\left|\frac{ax+b}{x}\right| + C$$

49) założenia: $abx(ax+b) \neq 0$

$$\int \frac{dx}{x^2(ax+b)}$$

Rozbijam funkcję podcalkowa na sume ulamków prostych :

$$\frac{1}{x^2(ax+b)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{ax+b}$$

$$1 \equiv Ax(ax+b) + B(ax+b) + Cx^2$$

podstawiam $x = 0$:

$$1 \equiv Bb \Rightarrow B = \frac{1}{b}$$

podstawiam $x = -\frac{b}{a}$

$$1 \equiv C \cdot \left(-\frac{b}{a}\right)^2 \Rightarrow C = \frac{a^2}{b^2}$$

Rozpisuj ę :

$$1 \equiv Ax(ax+b) + \frac{1}{b}(ax+b) + \frac{a^2}{b^2}x^2$$

$$1 \equiv Aax^2 + Abx + \frac{a}{b}x + 1 + \frac{a^2}{b^2}x^2$$

$$1 \equiv \left(Aa + \frac{a^2}{b^2}\right)x^2 + \left(Ab + \frac{a}{b}\right)x + 1$$

stąd:

$$0 = Aa + \frac{a^2}{b^2} \Rightarrow A = -\frac{a}{b^2}$$

Tak wiąa:

$$\int \frac{dx}{x^2(ax+b)} = -\frac{a}{b^2}\int \frac{dx}{x} + \frac{1}{b}\int \frac{dx}{x^2} + \frac{a^2}{b^2}\int \frac{dx}{ax+b} =< 21 >= -\frac{a}{b^2}\ln|x| - \frac{1}{bx} + \frac{a}{b^2}\ln|ax+b| + C =$$

$$-\frac{1}{bx} + \frac{a}{b^2}\ln\left|\frac{ax+b}{x}\right| + C$$

50) założenia: $abx(ax+b) \neq 0$

$$\int \frac{dx}{x^2(ax+b)^2}$$

Funkcję podcałkową rozbijam na sumę ułamków prostych.

$$\frac{1}{x^2(ax+b)^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{ax+b} + \frac{D}{(ax+b)^2}$$

$$1 \equiv Ax(ax+b)^2 + B(ax+b)^2 + Cx^2(ax+b) + Dx^2$$

podstawiam $x = 0$

$$1 \equiv Bb^2 \Rightarrow B = \frac{1}{b^2}$$

podstawiam $x = -\frac{b}{a}$

$$1 \equiv D \cdot \left(-\frac{b}{a}\right)^2$$

$$D = \frac{a^2}{b^2}$$

podstawiam:

$$1 \equiv Ax(ax+b)^2 + \frac{1}{b^2}(ax+b)^2 + Cx^2(ax+b) + \frac{a^2}{b^2}x^2$$

$$1 \equiv Ax(a^2x^2 + 2abx + b^2) + \frac{1}{b^2}(a^2x^2 + 2abx + b^2) + Cx^2(ax+b) + \frac{a^2}{b^2}x^2$$

$$1 \equiv Aa^2x^3 + 2Aabx^2 + Ab^2x + \frac{a^2}{b^2}x^2 + \frac{2a}{b}x + 1 + Cax^3 + Cbx^2 + \frac{a^2}{b^2}x^2$$

$$1 \equiv (Aa^2 + Ca)x^3 + \left(2Aab + \frac{2a^2}{b^2} + Cb\right)x^2 + \left(Ab^2 + \frac{2a}{b}\right)x + 1$$

a więw:

$$\begin{cases} Aa^2 + Ca = 0 \\ 2Aab + \dfrac{2a^2}{b^2} + Cb = 0 \\ Ab^2 + \dfrac{2a}{b} = 0 \end{cases}$$

z ostatniego równania wyliczam:

$$Ab^2 = -\frac{2a}{b}$$

$$A = -\frac{2a}{b^3}$$

podstawiam do rownania pierwszego:

$$Ca = \frac{2a^3}{b^3}$$

$$C = \frac{2a^2}{b^3}$$

tak więc:

$$\int \frac{dx}{x^2(ax+b)^2} = -\frac{2a}{b^3}\int \frac{dx}{x} + \frac{1}{b^2}\int \frac{dx}{x^2} + \frac{2a^2}{b^3}\int \frac{dx}{ax+b} + \frac{a^2}{b^2}\int \frac{dx}{(ax+b)^2} =< 21 >=$$

$$-\frac{2a}{b^3}\ln|x| - \frac{1}{b^2 x} + \frac{2a}{b^3}\ln|ax+b| + \frac{a}{b^2(ax+b)} + C = -a\left(\frac{1}{b^2(ax+b)} + \frac{1}{ab^2 x} - \frac{2}{b^3}\ln\left|\frac{ax+b}{x}\right|\right) + C$$

51) założenia: $x^2 - a^2 \neq 0$

$$\int \frac{dx}{x^2 - a^2} = \int \frac{dx}{(x-a)(x+a)}$$

rozkładam funkcję podcałkową na sumę ułamków prostych:

$$\frac{1}{(x-a)(x+a)} \equiv \frac{A}{x-a} + \frac{B}{x+a}$$

$$1 \equiv A(x+a) + B(x-a)$$

podstawiam $x = a$

$$1 \equiv 2Aa \Rightarrow A = \frac{1}{2a}$$

podstawiam $x = -a$

$$1 \equiv -2Ba \Rightarrow B = -\frac{1}{2a}$$

tak więc:

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\int \frac{dx}{x-a} - \frac{1}{2a}\int \frac{dx}{x+a} =< 21 >= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

52) założenia : $x^2 - a^2 \neq 0$

$$\int \frac{dx}{x^2 - a^2} = -\frac{1}{a^2}\int \frac{dx}{1-\left(\frac{x}{a}\right)^2} = \left\langle \begin{matrix} t = \frac{x}{a} \\ dt = \frac{dx}{a} \end{matrix} \right\rangle = \frac{1}{a}\int \frac{dt}{1-t^2} = \begin{cases} -\frac{1}{a}\operatorname{artgh} t + C; t \in (-1,1) \\ -\frac{1}{a}\operatorname{artgh} t + C; t \in (-\infty,-1)\cup(1,\infty) \end{cases} =$$

$$\begin{cases} -\frac{1}{a}\operatorname{artgh}\frac{x}{a} + C; x \in (-a,a) \\ -\frac{1}{a}\operatorname{artgh}\frac{x}{a} + C; x \in (-\infty,-1)\cup(1,\infty) \end{cases}$$

53) założenia: $a^2 - x^2 \geq 0$ a

$$\int x\sqrt{a^2 - x^2}\,dx = \left\langle \begin{matrix} t = a^2 - x^2 \\ dt = -2xdx \end{matrix} \right\rangle = -\frac{1}{2}\int t^{\frac{1}{2}}dt = -\frac{1}{3}t^{\frac{3}{2}} + C = -\frac{1}{3}\sqrt{\left(a^2 - x^2\right)^3} + C$$

54) $\int \cos^2 x dx = \int \frac{\cos 2x + 1}{2}dx = \frac{1}{2}\int \cos 2x dx + \frac{1}{2}\int dx =< 17 >= \frac{1}{4}\sin 2x + \frac{1}{2}x + C$

55) $\int \sin^2 x dx = \int \frac{\cos 2x - 1}{2}dx = \frac{1}{2}\int \cos 2x dx - \frac{1}{2}\int dx =< 17 > \tfrac{1}{4}\sin 2x - \frac{1}{2}x + C$

56) założenia : $a^2 - x^2 > 0$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \left\langle \begin{matrix} x = a\sin t \\ t = \arcsin\frac{x}{a} \\ dt = \frac{dx}{\sqrt{a^2 - x^2}} \end{matrix} \right\rangle = \int dt = t + C = \arcsin\frac{x}{a} + C$$

57) założenia :
$a^2 - x^2 > 0$

$$\int \sqrt{a^2 - x^2}\,dx = \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}}dx = \left\langle \begin{matrix} x = a\sin t \\ t = \arcsin\frac{x}{a} \\ dt = \frac{dx}{\sqrt{a^2 - x^2}} \end{matrix} \right\rangle = \int \left(a^2 - a^2\sin^2 t\right)dt = a^2\left(\int dt - \int \sin^2 t dt\right) =$$

$$\int a^2\left(\int dt - \int \frac{1-\cos 2t}{2}dt\right) = a^2\left(\int dt - \frac{1}{2}\int dt + \frac{1}{2}\int \cos 2t dt\right) = \left\langle \begin{matrix} u = 2t \\ du = 2du \end{matrix} \right\rangle = a^2\left(\frac{1}{2}\int dt + \frac{1}{4}\int \cos u du\right) =$$

$$a^2\left(\frac{1}{2}t + \frac{1}{4}\sin u\right) + C = a^2\left(\frac{1}{2}\arcsin\frac{x}{a} + \frac{1}{4}\sin\left(2\arcsin\frac{x}{2}\right)\right) + C =$$

$$a^2\left(\frac{1}{2}\arcsin\frac{x}{a} + \frac{1}{2}\sin\left(\arcsin\frac{x}{a}\right)\cos\left(\arcsin\frac{x}{a}\right)\right) + C = a^2\left(\frac{1}{2}\arcsin\frac{x}{a} + \frac{x}{2a}\sqrt{1-\left(\frac{x}{a}\right)^2}\right) + C$$

$$\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{ax}{2}\sqrt{1-\left(\frac{x}{a}\right)^2}+C=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C$$

$$\int\sqrt{x^2+a^2}\,dx=\int\frac{x^2+a^2}{\sqrt{x^2+a^2}}\,dx=\left\langle\begin{matrix}x=a\sinh t\\ t=\operatorname{arsinh}\frac{x}{a}\\ dt=\frac{dx}{\sqrt{x^2+a^2}}\end{matrix}\right\rangle=\int\left(a^2+a^2\sinh^2 t\right)dt=$$

58) $$a^2\int dt+a^2\int\sinh^2 t\,dt=\left\langle\sinh^2 x=\frac{\cosh 2x-1}{2}\right\rangle=a^2\int dt+\frac{a^2}{2}\int\cosh 2t\,dt-\frac{a^2}{2}\int dt=\left\langle\begin{matrix}u=2t\\ du=2dt\end{matrix}\right\rangle$$

$$\frac{a^2}{2}\int dt+\frac{a^2}{4}\int\cosh u\,du=\frac{a^2t}{2}+\frac{a^2}{4}\sinh(u)+C=\frac{a^2}{2}\operatorname{arsinh}\frac{x}{a}+\frac{a^2}{2}\cosh(\operatorname{arsinh}\frac{x}{a})\sinh\left(\operatorname{arsinh}\frac{x}{a}\right)+C=$$

$$\left\langle\begin{matrix}\sinh(\operatorname{arsinh}x)=x\\ \cosh(\operatorname{arsinh}x)=\sqrt{x^2+1}\end{matrix}\right\rangle=\frac{a^2}{2}\operatorname{arsinh}\frac{x}{a}+\frac{a^2}{2}\cdot\frac{x}{a}\cdot\sqrt{\frac{x^2}{a^2}+1}+C=\frac{a^2}{2}\operatorname{arsinh}\frac{x}{a}+\frac{x}{a}\sqrt{x^2+a^2}+C$$

59) $$\int\frac{dx}{\sqrt{a^2+x^2}}=\left\langle\begin{matrix}x=a\sinh t\\ t=\operatorname{arsinh}\frac{x}{a}\\ dt=\frac{dx}{\sqrt{a^2+x^2}}\end{matrix}\right\rangle=\int dt=t+C=\operatorname{arsinh}\frac{x}{a}+C$$

60) założenia: $a^2-x^2>0; x\neq 0$

$$\int\frac{\sqrt{a^2-x^2}\,dx}{x}=\int\frac{a^2-x^2}{x\sqrt{a^2-x^2}}\,dx=\left\langle\begin{matrix}x=a\sin t\\ t=\arcsin\frac{x}{a}\\ dt=\frac{dx}{\sqrt{a^2-x^2}}\end{matrix}\right\rangle=\int\frac{a^2-a^2\sin^2 t}{a\sin t}dt=a\int\frac{dt}{\sin t}-a\int\sin t\,dt$$

$$<30>=a\ln\left|\operatorname{tg}\frac{t}{2}\right|+a\cos t+C=<tg\frac{x}{2}=\frac{1-\cos x}{\sin x}>=a\ln\left|\frac{1-\cos t}{\sin t}\right|+a\cos t+C=$$

$$a\ln\left|\frac{1-\cos\left(\arcsin\frac{x}{a}\right)}{\sin\left(\arcsin\frac{x}{a}\right)}\right|+a\cos\left(\arcsin\frac{x}{a}\right)+C=<\begin{matrix}\sin(\arcsin x)=x\\ \cos(\arcsin x)=\sqrt{1-x^2}\end{matrix}>=a\ln\left|\frac{1-\sqrt{1-\frac{x^2}{a^2}}}{\frac{x}{a}}\right|+a\sqrt{1-\frac{x^2}{a^2}}+C=$$

$$a\ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right|+\sqrt{a^2-x^2}+C=-a\ln\left|\frac{x}{a-\sqrt{a^2-x^2}}\right|+\sqrt{a^2-x^2}+C=$$

$$\sqrt{a^2-x^2}-a\ln\left|\frac{x\left(a+\sqrt{a^2-x^2}\right)}{\left(a-\sqrt{a^2-x^2}\right)\left(a+\sqrt{a^2-x^2}\right)}\right|+C=\sqrt{a^2-x^2}-a\ln\left|\frac{x\left(a+\sqrt{a^2-x^2}\right)}{a^2-a^2+x^2}\right|+C=$$

$$\sqrt{a^2-x^2}-a\ln\left|\frac{x\left(a+\sqrt{a^2-x^2}\right)}{x^2}\right|+C=\sqrt{a^2-x^2}-a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|+C$$

61) $a^2-x^2>0; x\neq 0$

$$\int\frac{\sqrt{a^2-x^2}dx}{x}=\int\frac{x\sqrt{a^2-x^2}dx}{x^2}=\left\langle\begin{matrix}t=a^2-x^2\\ dt=-2xdx\\ x^2=a^2-t\end{matrix}\right\rangle=-\frac{1}{2}\int\frac{\sqrt{t}dt}{a^2-t}=\left\langle\begin{matrix}u^2=t\\ 2udu=dt\\ \sqrt{t}=u\end{matrix}\right\rangle=-\int\frac{u^2du}{a^2-u^2}=$$

$$\int\frac{-u^2+a^2-a^2}{a^2-u^2}du=\int du+a^2\int\frac{du}{u^2-a^2}=<51>=u+a^2\cdot\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C=u+\frac{a}{2}\ln\left|\frac{u-a}{u+a}\right|+C=$$

$$\sqrt{t}+\frac{a}{2}\ln\left|\frac{\sqrt{t}-a}{\sqrt{t}+a}\right|+C=\sqrt{a^2-x^2}+\frac{a}{2}\ln\left|\frac{\sqrt{a^2-x^2}-a}{\sqrt{a^2-x^2}+a}\right|+C$$

62) założenia: $a^2-x^2>0$

$$\int\frac{x^2dx}{\sqrt{a^2-x^2}}=\left\langle\begin{matrix}x=a\sin t\\ t=\arcsin\frac{x}{a}\\ dt=\frac{dx}{\sqrt{a^2-x^2}}\end{matrix}\right\rangle=\int a^2\sin^2 tdt=a^2\int\sin^2 tdt=<\sin^2 x=\frac{1-\cos 2x}{2}>=a^2\int\frac{1-\cos 2t}{2}dt=$$

$$\frac{a^2}{2}\int dt-\frac{a^2}{2}\int\cos 2tdt=\left\langle\begin{matrix}u=2t\\ du=2dt\end{matrix}\right\rangle=\frac{a^2}{2}\int dt-\frac{a^2}{4}\int\cos udu=\frac{a^2}{2}t-\frac{a^2}{4}\sin u+C=\frac{a^2}{2}t-\frac{a^2}{4}\sin 2t+C=$$

$$\frac{a^2}{2}t-\frac{a^2}{2}\sin t\cos t+C=\frac{a^2}{2}\arcsin\frac{x}{a}-\frac{a^2}{2}\sin\left(\arcsin\frac{x}{a}\right)\cos\left(\arcsin\frac{x}{a}\right)+C=<\begin{matrix}\sin(\arcsin x)=x\\ \cos(\arcsin)=\sqrt{1-x^2}\end{matrix}>=$$

$$\frac{a^2}{2}\arcsin\frac{x}{a}-\frac{a^2}{2}\cdot\frac{x}{a}\cdot\sqrt{1-\frac{x^2}{a^2}}+C=\frac{a^2}{2}\arcsin\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}+C$$

63) $\int x\sqrt{a^2+x^2}dx=\left\langle\begin{matrix}t=a^2+x^2\\ dt=2xdx\end{matrix}\right\rangle=\frac{1}{2}\int\sqrt{t}dt=\frac{1}{3}\sqrt{t^3}+C=\frac{1}{3}\sqrt{\left(a^2+x^2\right)^3}+C$

64) $\int x\sqrt{a^2+x^2}dx=\int\frac{x\left(a^2+x^2\right)dx}{\sqrt{a^2+x^2}}=\left\langle\begin{matrix}x=a\sinh t\\ t=\operatorname{arsinh}\frac{x}{a}\\ dt=\frac{dx}{\sqrt{a^2+x^2}}\end{matrix}\right\rangle=\int a\sinh t\left(a^2+a^2\sinh^2 t\right)dt=$

$$a^3\int\sinh tdt+a^3\int\sinh^3 tdt=<\sinh^2 x=\cosh^2 x-1>=a^3\int\sinh tdt+a^3\int\sinh t\left(\cosh^2 t-1\right)dt=$$

$$\left\langle \begin{matrix} u=\cosh t \\ du=\sinh t dt \end{matrix} \right\rangle = a^3\int \sinh t dt + a^3\int (u^2-1)du = a^3\cosh t + \frac{a^3}{3}u^3 - a^3 u + C =$$

$$\frac{a^3}{3}u^3 + C = \frac{a^3}{3}\cosh^3 t + C = \frac{a^3}{3}\cosh^3\left(\operatorname{arsinh}\frac{x}{a}\right) + C =< \cosh(\operatorname{arsinh} x) = \sqrt{1+x^2} >= \frac{a^3}{3}\sqrt{\left(1+\frac{x^2}{a^2}\right)^2} + C =$$

$$\frac{1}{3}\sqrt{(a^2+x^2)^3} + C$$

65)

$$\int \frac{xdx}{\sqrt{x^2+a^2}} = \left\langle \begin{matrix} t = x^2+a^2 \\ dt = 2xdx \end{matrix} \right\rangle = \frac{1}{2}\int \frac{2xdx}{\sqrt{x^2+a^2}} = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\int t^{-\frac{1}{2}}dt = \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = t^{\frac{1}{2}} + C =$$

$$\sqrt{a^2+x^2} + C$$

66)

$$\int \frac{xdx}{\sqrt{x^2+a^2}} = \left\langle \begin{matrix} x = a\sinh t \\ t = \operatorname{arsinh}\frac{x}{a} \\ dt = \frac{dx}{\sqrt{x^2+a^2}} \end{matrix} \right\rangle = \int a\sinh t dt = a\cosh t + C = a\cosh\left(\operatorname{arsinh}\frac{x}{a}\right) =$$

$$=< \cosh(\operatorname{arsinh} x) = \sqrt{x^2+1} >= a\sqrt{1+\frac{x^2}{a^2}} + C = \sqrt{x^2+a^2} + C$$

67) założenia: $x \neq 0$

$$\int \frac{dx}{\sinh x} = \int \frac{dx}{2\sinh\frac{x}{2}\cosh\frac{x}{2}} = \int \frac{dx}{2\operatorname{tgh}\frac{x}{2}\cosh\frac{x}{2}\cosh\frac{x}{2}} = \int \frac{dx}{2\operatorname{tgh}\frac{x}{2}\cosh^2\frac{x}{2}} = \left\langle \begin{matrix} t = \operatorname{tgh}\frac{x}{2} \\ dt = \frac{dx}{2\cosh^2\frac{x}{2}} \end{matrix} \right\rangle =$$

$$\int \frac{dt}{t} = \ln|t| + C = \ln\left|\operatorname{tgh}\frac{x}{2}\right| + C$$

68) $\int \sinh cxdx = \frac{1}{c}\int \sinh cx \cdot cdx = \left\langle \begin{matrix} t = cx \\ dt = cdx \end{matrix} \right\rangle = \frac{1}{c}\int \sinh tdt = \frac{1}{c}\cosh t + C = \frac{1}{c}\cosh cx + C$

69) $\int \cosh cxdx = \frac{1}{c}\int \cosh cx \cdot cdx = \left\langle \begin{matrix} t = cx \\ dt = cdx \end{matrix} \right\rangle = \frac{1}{c}\int \cosh tdt = \frac{1}{c}\sinh t + C = \frac{1}{c}\cosh cx + C$

70) $\int \sinh^2 xdx = \left\langle \sinh^2 x = \frac{\cosh 2x - 1}{2} \right\rangle = \int \frac{\cosh 2x - 1}{2}dx = \frac{1}{2}\int \cosh 2x - \frac{1}{2}\int dx =< 69 > \frac{1}{4}\sinh 2x - \frac{1}{2}x + C$

71)

$$\int \cosh^2 x dx = \left\langle \cosh^2 x = \frac{\cosh 2x + 1}{2} \right\rangle = \int \frac{\cosh 2x+1}{2} dx = \frac{1}{2}\int \cosh 2x dx + \frac{1}{2}\int dx =<69>=$$

$$\frac{1}{4}\sinh 2x + \frac{1}{2}x + C$$

Spis treści:

$\frac{1}{(x-k)^2+b}$	$\frac{1}{\sqrt{b}} arctg \frac{x-k}{\sqrt{b}}$	$b \in R_+$	**29**
$x(ax+b)^n$	$\frac{a(n-1)x-b}{a^2(n+1)(n+2)}(ax+b)^{n+1}$	$n \notin \{-2,-1\}$	**40**
$\frac{x}{ax+b}$	$\frac{x}{a}+\frac{b}{a^2}-\frac{b}{a^2}\ln\lvert ax+b\rvert+C$	$ax+b \neq 0$	**41**
$\frac{x}{(ax+b)^2}$	$\frac{1}{a^2(ax+b)}+\frac{1}{a^2}\ln\lvert ax+b\rvert$	$ax+b \neq 0$	**42**
$\frac{x}{(ax+b)^n}$	$\frac{a(1-n)x-b}{a^2(n-1)(n-2)(ax+b)^{n-1}}$	$ax+b \neq 0; n \notin \{1,2\}$	**43**
$\frac{x^2}{ax+b}$	$\frac{1}{a^3}\left[\frac{(ax+b)^2}{2}-2b(ax+b)+b^2\ln\lvert ax+b\rvert\right]$	$ax+b \neq 0$	**44**
$\frac{x^2}{(ax+b)^2}$	$\frac{1}{a^3}\left[ax+b-2b\ln\lvert ax+b\rvert-\frac{b^2}{ax+b}\right]$	$ax+b \neq 0$	**45**
$\frac{x^2}{(ax+b)^3}$	$\frac{1}{a^3}\left[\ln\lvert ax+b\rvert+\frac{2b}{ax+b}-\frac{b^2}{2(ax+b)^2}\right]$	$ax+b \neq 0$	**46**
$\frac{x^2}{(ax+b)^n}$	$\frac{1}{a^3}\left[-\frac{1}{(n-3)(ax+b)^{n-3}}+\frac{2b}{(n-2)(ax+b)^{n-2}}-\frac{b^2}{(n-1)(ax+b)^{n-1}}\right]$	$ax+b \neq 0; n \notin \{1,2,3\}$	**47**
$\frac{1}{x(ax+b)}$	$-\frac{1}{b}\ln\left\lvert\frac{ax+b}{x}\right\rvert$	$x(ax+b) \neq 0; b \neq 0$	**48**
$\frac{1}{x^2(ax+b)}$	$-\frac{1}{bx}+\frac{a}{b^2}\ln\left\lvert\frac{ax+b}{x}\right\rvert$	$abx(ax+b) \neq 0$	**49**
$\frac{1}{x^2(ax+b)^2}$	$-a\left(\frac{1}{b^2(ax+b)}+\frac{1}{ab^2x}-\frac{2}{b^3}\ln\left\lvert\frac{ax+b}{x}\right\rvert\right)$	$abx(ax+b) \neq 0$	**50**
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$	$x^2-a^2 \neq 0$	**51**
$\frac{1}{x^2-a^2}$	$\begin{cases} -\frac{1}{a}\text{artgh}\frac{x}{a}+C; x \in (-a,a) \\ -\frac{1}{a}\text{arctg}\frac{x}{a}+C; x \in (-\infty,-1)\cup(1,\infty) \end{cases}$	$x^2-a^2 \neq 0$	**52**
Całki funkcji niewymiernych			
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\frac{x}{a}$	$a^2-x^2>0$	**9**
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\frac{x}{a}$	$a^2-x^2>0$	**56**
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}$	$a^2-x^2>0$	**10**
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}$	$a^2-x^2>0$	**57**

$\frac{1}{\sqrt{x^2+k}}$	$\ln\lvert x+\sqrt{x^2+k}\rvert$	$k>0$	**11**
$\frac{1}{\sqrt{x^2+a^2}}$	$\operatorname{arsinh}\frac{x}{a}$		**59**
$\sqrt{x^2+k}$	$\ln\lvert x+\sqrt{x^2+k}\rvert+\frac{x}{2}\sqrt{x^2+k}$	$k>0$	**12**
$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\operatorname{arsinh}\frac{x}{a}+\frac{x}{a}\sqrt{x^2+a^2}$		**58**
$\sqrt{a+bx}$	$\frac{2}{3b}\sqrt{(a+bx)^3}$	$a,b\in R_+$	**23**
$x\sqrt{a^2-x^2}$	$-\frac{1}{3}\sqrt{\left(a^2-x^2\right)^3}$	$a^2-x^2\geq 0$	**53**
$\frac{\sqrt{a^2-x^2}}{x}$	$\sqrt{a^2-x^2}-a\ln\left\lvert\frac{a+\sqrt{a^2-x^2}}{x}\right\rvert$	$a^2-x^2>0; x\neq 0$	**60**
$\frac{\sqrt{a^2-x^2}}{x}$	$\frac{a}{2}\ln\left\lvert\frac{\sqrt{a^2-x^2}-a}{\sqrt{a^2-x^2}+a}\right\rvert$	$a^2-x^2>0; x\neq 0$	**61**
$\frac{x^2}{\sqrt{a^2-x^2}}$	$\frac{a^2}{2}\arcsin\frac{x}{a}-\frac{x}{2}\sqrt{a^2-x^2}$	$a^2-x^2>0$	**62**
$x\sqrt{a^2+x^2}$	$\frac{1}{3}\sqrt{\left(a^2+x^2\right)^3}$		**63**
$x\sqrt{a^2+x^2}$	$\frac{1}{3}\sqrt{\left(a^2+x^2\right)^3}$		**64**
$\frac{x}{\sqrt{x^2+a^2}}$	$\sqrt{x^2+a^2}$		**65**
$\frac{x}{\sqrt{x^2+a^2}}$	$\sqrt{x^2+a^2}$		**66**
Całki funkcji trygonometrycznych			
$\operatorname{tg} x$	$-\ln\lvert\cos x\rvert$	$x\in R\setminus\bigcup_{k\in Z}\left\{k\pi+\frac{\pi}{2}\right\}$	**2**
$\operatorname{ctg} x$	$\ln\lvert\sin x\rvert$	$x\in R\setminus\bigcup_{k\in Z}\left\{k\pi\right\}$	**3**
$\sin^n x$	$-\frac{1}{n}\cos x\sin^{n-1}x+\frac{n-1}{n}\int\sin^{n-2}xdx$	$n\in Z\cap(3,\infty)$	**7**
$\cos^n x$	$\frac{1}{n}\sin x\cos^{n-1}x+\frac{n-1}{n}\int\cos^{n-2}xdx$	$n\in Z\cap(3,\infty)$	**8**
$\sin mx\sin nx$	$\frac{\sin[x(m-n)]}{2(m-n)}-\frac{\sin[x(m+n)]}{2(m+n)}$	$m\neq\pm n$	**14**
$\cos mx\cos nx$	$\frac{\sin[x(m+n)]}{2(m+n)}+\frac{\sin[x(m-n)]}{2(m-n)}$	$m\neq\pm n$	**15**

$\sin mx\cos nx$	$-\frac{\cos[x(m+n)]}{2(m+n)}-\frac{\cos[x(m-n)]}{2(m+n)}$	$m\neq\pm n$	**16**
$\cos mx$	$\frac{\sin mx}{m}$	$m\neq 0$	**17**
$\sin mx$	$-\frac{\cos mx}{m}$	$m\neq 0$	**18**
$tgmx$	$\frac{\ln\lvert\cos mx\rvert}{m}$	$m\neq 0$	**19**
$ctgmx$	$\frac{\ln\lvert\sin mx\rvert}{m}$	$m\neq 0$	**20**
$\frac{\sin x}{a+b\cos x}$	$-\frac{\ln\lvert a+b\cos x\rvert}{b}$	$b\neq 0; a+b\cos x\neq 0$	**22**
$\frac{1}{\sin x}$	$\ln\left\lvert\operatorname{tg}\frac{x}{2}\right\rvert$	$\sin x\neq 0$	**30**
$\frac{1}{\cos x}$	$\ln\lvert\operatorname{tg}(\frac{\pi}{4}+\frac{x}{2})\rvert$	$\cos x\neq 0$	**31**
$\frac{1}{\sin x\cos x}$	$\ln\lvert\operatorname{tg}x\rvert$	$\sin x\cos x\neq 0$	**32**
$\frac{1}{\sin^2 x\cos^2 x}$	$\operatorname{tg}x-\operatorname{ctg}x$	$\sin x\cos x\neq 0$	**33**
$\operatorname{tg}^2 x$	$\operatorname{tg}x-x$	$\in R\setminus\bigcup_{k\in Z}\left\{\frac{\pi}{2}+k\pi\right\}$	**34**
$\operatorname{ctg}^2 x$	$-\operatorname{ctg}x-x$	$x\in R\setminus\bigcup_{k\in Z}\left\{k\pi\right\}$	**35**
$\operatorname{tg}^n x$	$\frac{1}{n-1}\operatorname{tg}^{n-1}-\int\operatorname{tg}^{n-2}xdx$	$n\in Z; n>2; x\in R\setminus\bigcup_{k\in Z}\left\{\frac{\pi}{2}+k\pi\right\}$	**36**
$\operatorname{ctg}^n x$	$-\frac{1}{n-1}\operatorname{ctg}^{n-1}x-\int\operatorname{ctg}^{n-2}xdx$	$n\in Z; n>2; x\in R\setminus\bigcup_{k\in Z}\left(k\pi\right\}$	**37**
$x\sin cx$	$\frac{\sin cx}{c^2}-\frac{x\cos x}{c}$		**38**
$\frac{1}{\sin cx}$	$\frac{1}{c}\ln\lvert\operatorname{tg}\frac{cx}{2}\rvert$	$\sin cx\neq 0$	**39**
$\cos^2 x$	$\frac{1}{4}\sin 2x+\frac{1}{2}x$		**54**
$\sin^2 x$	$\frac{1}{4}\sin 2x-\frac{1}{2}x$		**55**
Całki funkcji hiperbolicznych			
$\sinh cx$	$\frac{1}{c}\cosh cx$		**68**
$\cosh cx$	$\frac{1}{c}\sinh cx$		**69**
$\sinh^2 x$	$\frac{1}{4}\sinh 2x-\frac{1}{2}x$		**70**

$\cosh^2 x$	$\frac{1}{4}\sinh 2x + \frac{1}{2}x$		**71**
$\frac{1}{\sinh x}$	$\ln\left\lvert \operatorname{tgh}\frac{x}{2}\right\rvert$	$x \neq 0$	**67**
Całki funkcji wykładniczych			
Całki funkcji logarytmicznych			
$\ln x$	$x(\ln x - 1)$	$x \in R_+$	**4**
Całki funkcji arcus			
Całki funkcji area			
Wzory rekurencyjne			
$\sin^n x$	$-\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx$	$n \in Z \cap (3, \infty)$	**7**
$\cos^n x$	$\frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x dx$	$n \in Z \cap (3, \infty)$	**8**
$\frac{1}{(x^2+1)^n}$	$\frac{1}{2n-2} \cdot \frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\int \frac{dx}{(x^2+1)^{n-1}}$	$n \in Z; n > 1$	**26**
$\operatorname{tg}^n x$	$\frac{1}{n-1}\operatorname{tg}^{n-1} - \int \operatorname{tg}^{n-2} x dx$	$n \in Z; n > 2; x \in R \setminus \bigcup_{k \in Z}\left\{\frac{\pi}{2} + k\pi\right\}$	**36**
$\operatorname{ctg}^n x$	$-\frac{1}{n-1}\operatorname{ctg}^{n-1} x - \int \operatorname{ctg}^{n-2} x dx$	$n \in Z; n > 2; x \in R \setminus \bigcup_{k \in Z}\left(k\pi \right\}$	**37**
Inne			
$\frac{f'(x)}{f(x)}$	$\ln\lvert f(x)\rvert$	$f(x) \neq 0$	**13**